# An Epistemic Extension of Threshold Models: Coordination based on Behavior Prediction


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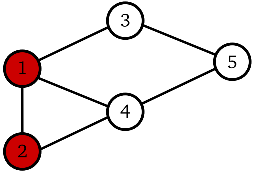
Modality and Modalities
Lund, May 22, 2014

# Rational Adoption: Coordination Games on Networks

## Threshold model: $\mathcal{M} = (\mathcal{A}, N, B, \theta)$

$(\mathcal{A}, N)$ is a network, $B \subseteq \mathcal{A}$ a behavior and $\theta \in [0,1]$ a uniform adoption threshold. Models are updated by

$$B_{n+1} := B_n \cup \{i \in \mathcal{A} : \frac{N(i) \cap B_n}{N(i)} \geq \theta\}.$$



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These dynamics are equivalent to the best response dynamics of agents playing a coordination game with their neighbors:

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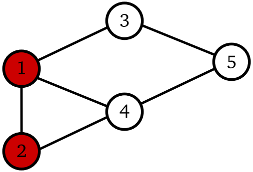
| $\overline{B}$ | $0, 0$ | $y, y$ |

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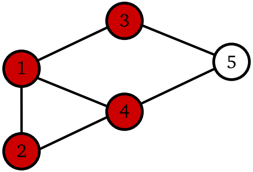


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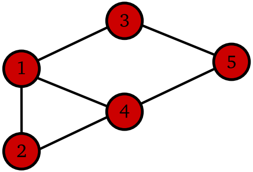


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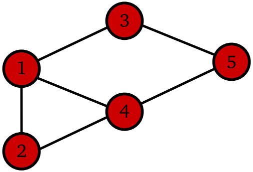


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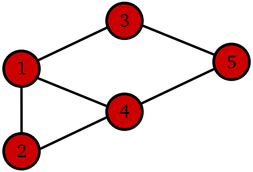


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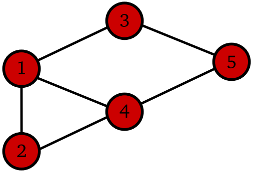


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Implicit epistemic assumption: agents only know the behavior of their neighbors.

# Outline

– Add an epistemic dimension

– Updates of epistemic threshold models according to normal update dynamics

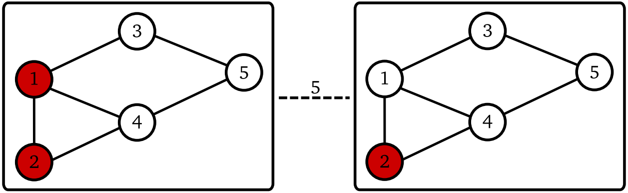
– Define an update where agents predicts the behavior of agents one level lower than themselves

– Explain prediction update by some results

– Further research

# Adding Epistemics

An epistemic dimension is added by using threshold models as epistemic alternatives:



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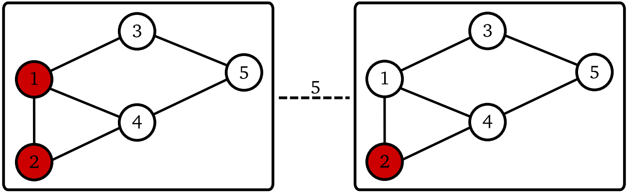


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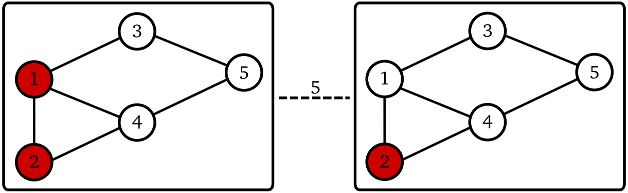


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## Epistemic Threshold Model with sight $k$: $\mathbb{M} = (|\mathbb{M}|, \{\sim_i\}_{i \in \mathcal{A}})$

$|\mathbb{M}|$ is a set of threshold models s.t. $\forall \mathcal{M}, \mathcal{M}' \in |\mathbb{M}|$,
if $\mathcal{M} = (\mathcal{A}, N, B, \theta)$ and $\mathcal{M}' = (\mathcal{A}', N', B', \theta')$, then $\mathcal{A} = \mathcal{A}'$, $N = N'$ and $\theta = \theta'$.

$\sim_i$ is an equivalence relation on $|\mathbb{M}|$ s.t. if $\mathcal{M} \sim_i \mathcal{M}'$, then

$$\forall j \in N^k(i) \cup \{i\} : j \in B_{\mathcal{M}} \Leftrightarrow j \in B'_{\mathcal{M}'}$$

where $N^k(i)$ is the set of $k$-reachable neighbors of $i$.

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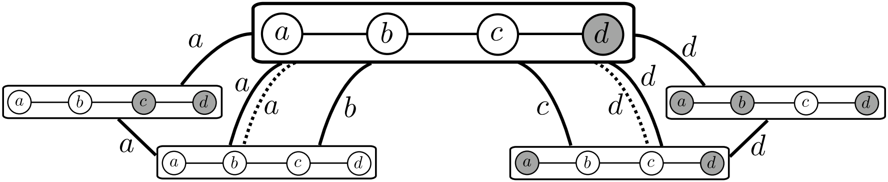


—: sight 1, $\cdots$: sight 2.

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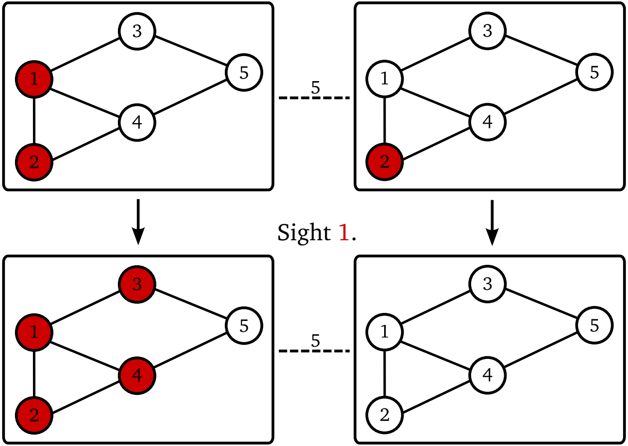
ETMs may be updated using the previous update, but we must also update the $\sim_i$'s:



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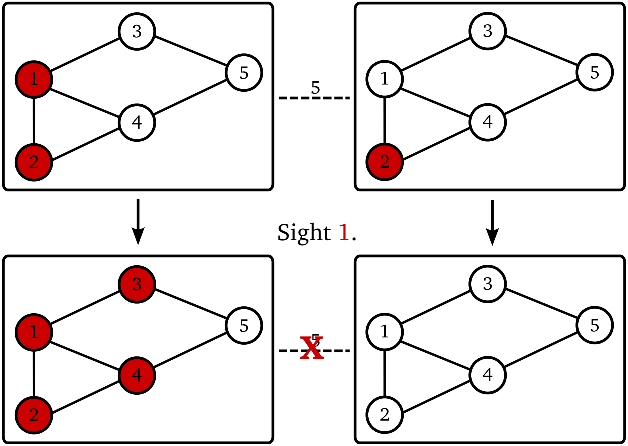


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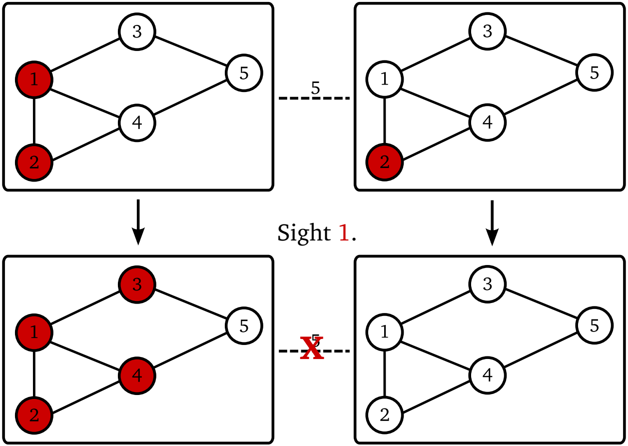



Simply restrict each $\sim_i$ to satisfy the requirement from the definition of ETMs.

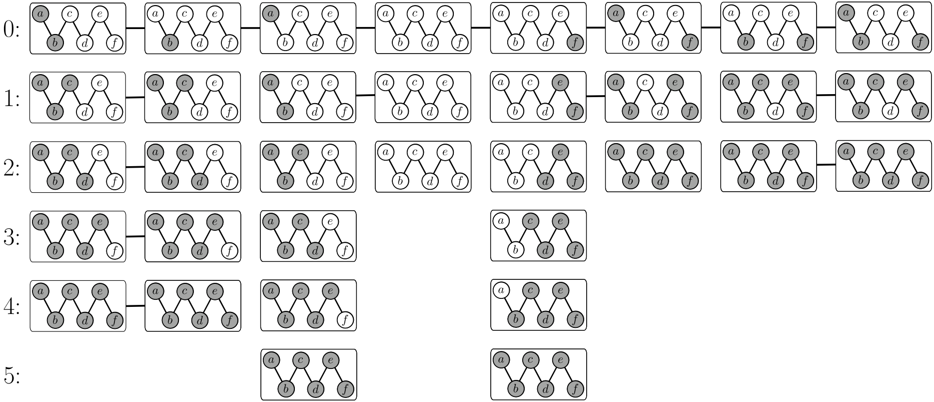
This doesn't get 5 any better off, though.

# Updating Epistemic Threshold Modes: Learning

Restricting $\sim_i$'s allow agents to learn about the initial configuration.

Example with sight 1, and only $\sim_d$ depicted.

Two states are connected iff the behaviors of $c$ and $e$ are identical.



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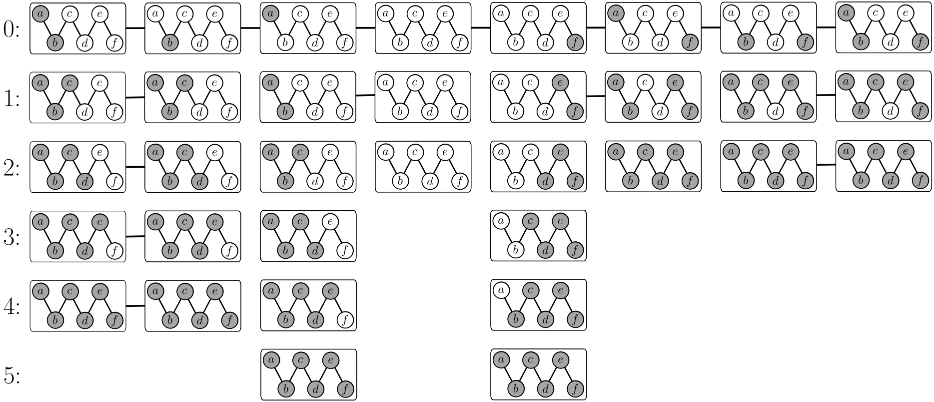
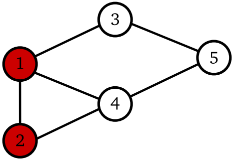


Depending on the actual state, $d$'s learning may or may not be complete.

# Predicting Behavior




To help 5 out, we endow her with the power of prediction:

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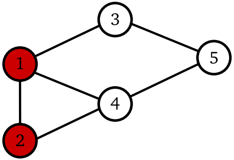


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## $k$-level Prediction Update

Given $\mathbb{M}_n$ and $B_n$ from $\mathcal{M} \in |\mathbb{M}_n|$, the $k$-level prediction update of $\mathbb{M}_n$ produces $\mathbb{M}_{n+1}$, identical to $\mathbb{M}_n$ except that
– The $k$-level prediction update of $B_n$ is given by

$$B^k_{n+1} := B_n \cup \{a \in \mathcal{A} : \frac{|N(a) \cap K_a B^{k-1}_{n+1}|}{|N(a)|} \geq \theta\}$$

where $K_a B^{k-1}_{n+1}$ is the set of agents s.t. $a$ knows that if these agents updated in accordance with $k-1$ level prediction update, then they will adopt in the next round:

$$K_a B^{k-1}_{n+1} := \{j \in \mathcal{A} : \forall \mathcal{M}' \sim_a \mathcal{M}, j \in B^{k-1}_{n+1}\}$$

with $B^0_{n+1}$ the behavior set obtained if blind adopt update is applied to $\mathcal{M}$.
– All relations $\sim_i$ are restricted to satisfy the requirement for ETMs.

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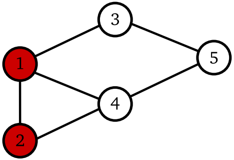


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Prediction level

Time

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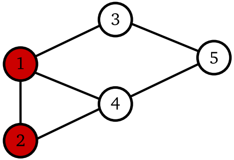


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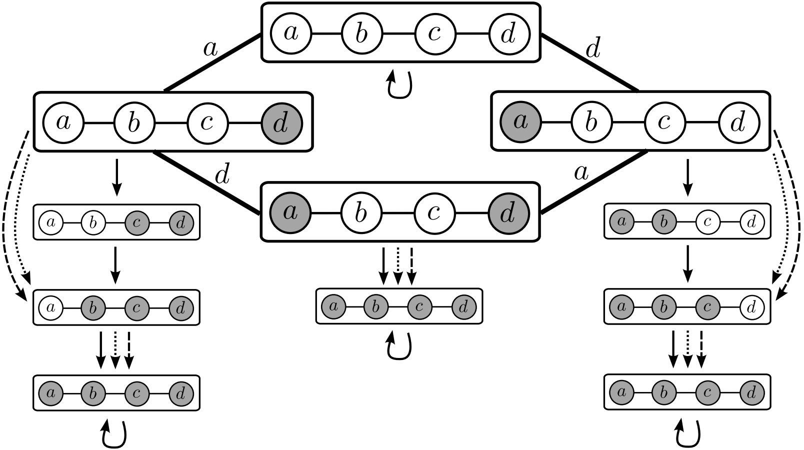
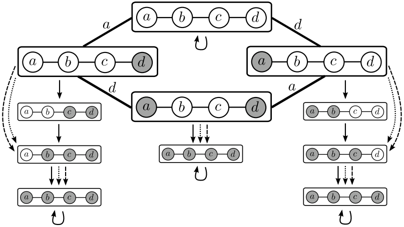
# Prediction Update: Example




Prediction update on an epistemic threshold model with $\theta \leq \frac{1}{2}$ and sight 2.

Full arrows show transitions for level 0 prediction, dotted arrows for level 1 and dashed arrows level 2.

# Results




## Proposition

Let $\mathbb{M}$ be a $k$-sight epistemic threshold model with actual world $\mathcal{M} = (\mathcal{A}, N, B, \theta)$. Then

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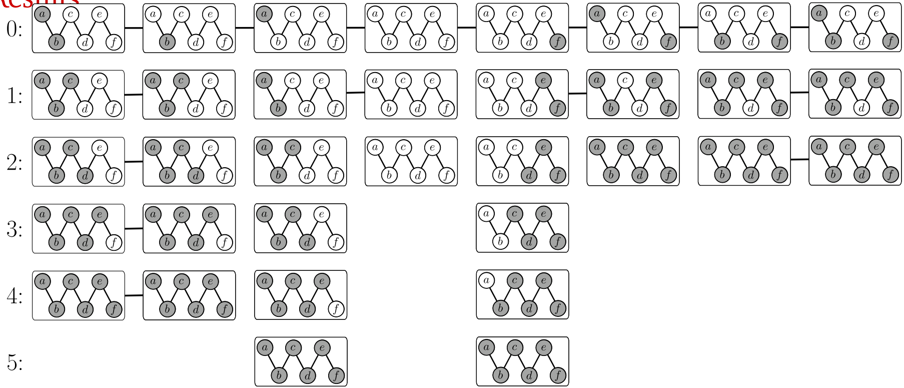

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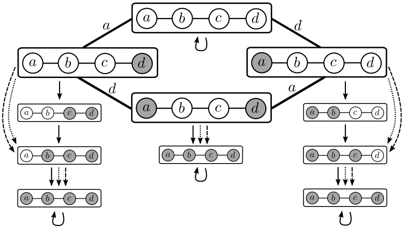

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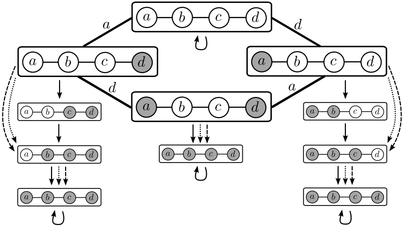


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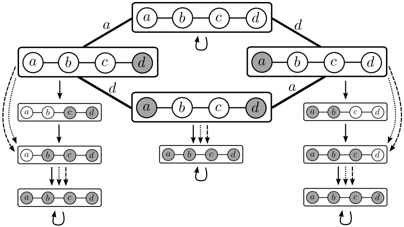
3. Increased prediction does not slow dynamics: for $m \geq k$, $B_n^k \subseteq B_n^m$
4. Knowledge is does not diminish with prediction level: for $m \geq k$, $K_b B_n^k \subseteq K_b B_n^m$.

# Main Result 1




## Theorem 1

All prediction dynamics are fixed point equivalent to blind adoption dynamics. Specifically, for all $k$, if $B_n^k = B_{n+1}^k$ and $B_m^0 = B_{m+1}^0$, then $B_n^k = B_m^0$.

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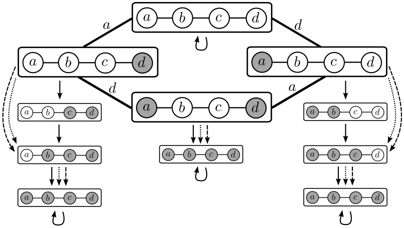


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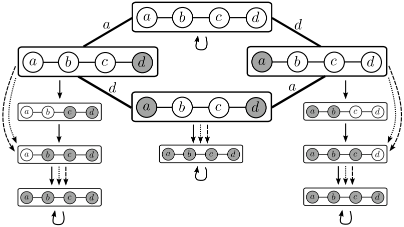
## Corollary

The Cluster Theorem for the standard threshold update also applies to prediction updates.

# Main Result 2




## Theorem 2

Prediction is limited by sight. If $\sim_i$ is defined using $N^k(i)$, then if $m \geq k-1$, then $B_n^m = B_n^{k-1}$, for all $n$.

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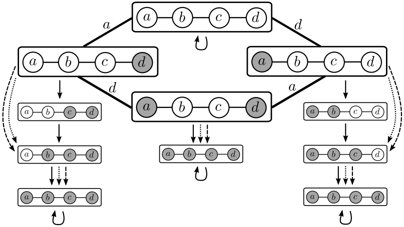


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Theorem 2 provides us with an epistemic characterization of the standard dynamics:

## Corollary

Prediction dynamics are step-wise equivalent to the standard dynamics exactly in epistemic threshold models with sight $1$.

# Further research

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- Model anticipation: agents currently predict the standard dynamics, but do not seek to influence them. We would like to define trend-setting reasoning: “If I change my behavior prematurely, then those guys will follow – and then I will benefit”.

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- Fixed point definition of prediction update
- Defining a suitable dynamic logic
- Model anticipation: agents currently predict the standard dynamics, but do not seek to influence them. We would like to define trend-setting reasoning: “If I change my behavior prematurely, then those guys will follow – and then I will benefit”.
- Generalizations of the epistemic models:
  - Drop common knowledge of network and thresholds
  - Work with weighted, directed graphs