Modal correspondence theory over presheaves

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Modality and Modalities
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Outline

1. A categorical notion of bisimulation
2. Example: **Bran**$_L$
3. Path Logic
4. Correspondence

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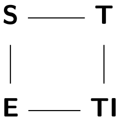

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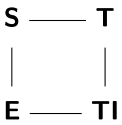

Each model has its own notion of behavioural equivalence. Is there a **unifying categorical notion of bisimulation**?

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   - objects of $\mathbf{P}$ are a path-shapes
   - an arrow $m : P \to P'$ in $\mathbf{P}$ tells us that P' extends P
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4. see objects of $\mathbf{M}$ as presheaves over $\mathbf{P}$

P-open maps

Definition

An arrow $f : M \to M'$ in $\mathbf{M}$ is $\mathbf{P}$-*open* if for any morphism $m : P \to P'$ in $\mathbf{P}$ and commuting square

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there exists $r : P' \to M$ such that the triangles commute in

$$\begin{array}{ccc} P & \xrightarrow{p} & M \\ {\scriptstyle m}\downarrow & \overset{r}{\nearrow} & \downarrow{\scriptstyle f} \\ P' & \xrightarrow[q]{} & M' \end{array}$$

Path bisimulations

Suppose **P** has an initial object I:

Definition

A *path bisimulation* between models of concurrency M and M' over a path category **P** is a set Z of pairs of paths (p_1, p_2) such that $p_1 : P \to M$ and $p_2 : P \to M'$ satisfying the following conditions:

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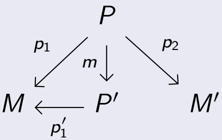

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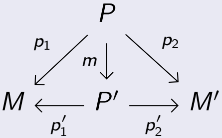

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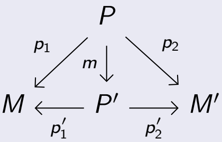

3. the condition symmetric to 2, plus other two conditions where m is reversed

Presheaf representation

We can see a model of concurrency as a presheaf $\mathbf{P}^{op} \to \mathbf{Set}$ using a Yoneda-like construction:

$$M \mapsto F = Hom_{\mathbf{M}}(-, M)$$

Conceptual idea: we can see a model of concurrency as a collection of paths glued together. Moreover, while $m : P \to P'$ tells us how P' extends P, $F(m) : F(P') \to F(P)$ indicates how paths of sort P' restrict to paths of sort P.

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Say $\mathbf{P}$ has an initial object I, an *empty path*.

Definition

A presheaf is *rooted* if the image of I is a singleton.

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Definition

Call $\mathbf{T}_L$ the category of pointed transition systems with labels L, where morphisms are preserve transition and initial states. Call $\mathbf{Tree}_L$ the subcategory of $\mathbf{T}_L$ consisting of trees. Call $\mathbf{Bran}_L$ (a skeleton of) the subcategory consisting of only branches, i.e. linear paths.

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Following Nielsen and Winskel we can encode pointed transition systems with labels L into the presheaf category $\mathbf{Set}^{\mathbf{Bran}_L^{op}}$ ([2], [4]):

$$\begin{array}{ccc} \mathbf{T}_L & \xrightarrow{Pre} & rooted(\mathbf{Set}^{\mathbf{Bran}_L^{op}}) \\ & {\scriptstyle u}\searrow & \downarrow{\scriptstyle \simeq} \\ & & \mathbf{Tree}_L \end{array}$$

Everything works fine for transition systems:

Theorem ([2])

When $\mathbf{M}$ *is* $\mathbf{T}_L$ *and* $\mathbf{P}$ *is* $\mathbf{Bran}_L$ *we have that:*

- $\mathbf{P}$*-open maps are p-morphisms*
- *given two transition systems* M_1 *and* M_2*, TFAE:*
  1. M_1 *and* M_2 *are bisimilar*
  2. *there is a path-bisimulation between* M_1 *and* M_2
  3. *there is a span of* $\mathbf{P}$*-open between* M_1 *and* M_2

Outline

The path logic $\mathbb{PL}_{\mathbf{P}}$

The logic that is characteristic for path-bisimulation is ([2]):

$$\varphi ::= \quad \bot \mid \neg\varphi \mid \bigwedge_{j \in J} \varphi_j \mid \langle m \rangle \varphi \mid \overline{\langle m \rangle} \varphi$$

where m is a morphism in $\mathbf{P}$ and J has cardinality $max\{|\mathbf{P}(X, Y)| | X, Y \in \mathbf{P}_0\}$.

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where m is a morphism in $\mathbf{P}$ and J has cardinality $max\{|\mathbf{P}(X,Y)| | X, Y \in \mathbf{P}_0\}$.
Given an object P of $\mathbf{P}$, an object M of $\mathbf{M}$ and a path $p : P \to M$:
$(P, p) \vDash \langle m \rangle \varphi \quad$ iff there exists p' s.t.

$$\begin{array}{ccc} P & \xrightarrow{m} & P' \\ \downarrow p & \swarrow & p' \vDash \varphi \\ M & & \end{array}$$

for the condition of $\overline{\langle m \rangle}\varphi$ switch p and p'.

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If $\mathbf{P}$ is small, $F : \mathbf{P}^{op} \to \mathbf{Set}$ can be made into a relational structure:

- $W = \{(P, x) | P \in \mathbf{P}_0, x \in F(P)\}$
- for every pair of objects (P, x) and (P', x') define $(P, x) R_m (P', x')$ iff

$$\begin{array}{ccc} P & \xrightarrow{m} & P' \\ | & & | \\ x & \underset{F(m)}{\longleftarrow} & x' \end{array}$$

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Correspondence theory for path logics

It is also natural to consider the finitary path logic:

$$\varphi ::= \quad \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle m \rangle \varphi \mid \overline{\langle m \rangle} \varphi$$

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- Our yardstick will be a multi-sorted first-order language, interpreted over presheaves.

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The first-order language of a path category

Given a small path category **P**, we may treat the objects of **P** as *sorts*, we introduce a binary predicate R_m of for each morphism m and a countably infinite supply of variables of each sort.
The syntax of the multi-sorted language is given by:

$$\varphi ::= x_P R_m y_Q \mid x_P = y_P \mid \neg\varphi \mid \varphi \wedge \varphi \mid \exists x_P \varphi$$

Here, $m : P \to Q$. Then

$$\begin{aligned} F \vDash_{\upsilon} x_P R_m y_Q &\Leftrightarrow (P, x) R_m (Q, x') \\ &\Leftrightarrow Fm(\upsilon(y_Q)) = \upsilon(x_P) \end{aligned}$$

Our main observation

Suppose a path category $\mathbf{P}$ has an initial object I. Call *rooted* the presheaves assigning a singleton to I.

Theorem

For any such $\mathbf{P}$, a formula of the associated multi-sorted FOL open in the variable x_I is invariant for path bisimulations on rooted $\mathbf{P}$-presheaves iff it is equivalent to a formula of the path logic for $\mathbf{P}$.

Proof sketch

- We can interpret both the multi-sorted first-order logic and path logic over multi-sorted models, and it makes sense to speak of bisimulations between such models in general.

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- Due to a mild adaptation of a known observation from modal correspondence theory ([1]), a correspondence theorem can be proved essentially using the same methods as in van Benthem's original correspondence theorem relative to any *elementary* class of models.
- So given **P**, all we need to show is that the class of multi-sorted models corresponding to rooted presheaves over **P** is elementary!

Axioms for rooted presheaves

1. $\exists! x_I (x_I = x_I)$
2. Given a path $h : C \to D$: $\forall y_D \exists! x_C (x_C R_h y_D)$
3. For any object C: $\forall x_C \forall y_C (x_C R_{Id_C} y_C \leftrightarrow x_C = y_C)$
4. Given paths $f : A \to B$ and $g : B \to C$:

$$\forall x_A \forall z_C (x_A R_{(g \circ f)} z_C \leftrightarrow \exists y_B (x_A R_f y_B \wedge y_B R_g z_C))$$

The case of $\mathbf{Bran}_L$

Rooted presheaves over $\mathbf{Bran}_L$ are essentially labelled trees. Interestingly, van Benthem's theorem, in its classical formulation, is false restricted to trees:

$$\exists x \forall y \neg (x R_a y)$$

This formula is bisimulation invariant over trees, but not equivalent to any basic modal formula. Over L-labelled trees, it corresponds to the μ-calculus formula

$$\mu Z. \bigvee_{b \in L} \langle b \rangle Z \vee [a] \perp$$

Decidability of the equivalence with path formulas

It is also known that it is undecidable whether a given formula of (one-sorted) FOL is equivalent to a formula of basic modal logic ([3]). By constrast, we have the following result:

Theorem

There is an effective procedure to determine, for a given formula of the multi-sorted FOL for $\mathbf{Bran}_L$, whether this formula is equivalent to any formula of the path logic for $\mathbf{Bran}_L$ over L-trees/$\mathbf{Bran}_L$-presheaves.

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- Every quantified variable has a sort P in $\mathbf{Bran}_L$, and every such sort P refers to a "path shape" of a fixed finite length.

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- Any tree of finite height can be described up to bisimilarity by a single path logic formula of corresponding modal depth, and so any bisimulation invariant FOL formula is equivalent to a disjunction of such formulas.

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- Any tree of finite height can be described up to bisimilarity by a single path logic formula of corresponding modal depth, and so any bisimulation invariant FOL formula is equivalent to a disjunction of such formulas.
- Since the number of formulas up to equivalence of any given modal depth is effectively bounded, there is a computable number of "candidates" for equivalent modal formulas. This gives a reduction to the satisfiability problem for multi-sorted FOL over L-trees/$\mathbf{Bran}_L$-presheaves.

Decidability of multi-sorted FOL over L-trees

Theorem

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Proof sketch:

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- Encode the existence of a suitable tree in a monadic second order formula. The results follows from the decidability of $S\omega S$, the theory of the ω-branching tree in MSO.

Questions:

1. What properties of the path category **P** are required to obtain decidable correspondence theorems?
2. For what path categories **P** are the associated first-order logics decidable?
3. Is there a syntactic criterion to recognize the multi-sorted formulas that correspond to path logic formulas?
4. Applications of path logic in other contexts where “open maps” appear, topology, topos theory etc.?

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