A Minimal Dynamic Logic for Threshold Influence

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Joint work with Alexandru Baltag, Rasmus K. Rendsvig and Sonja Smets

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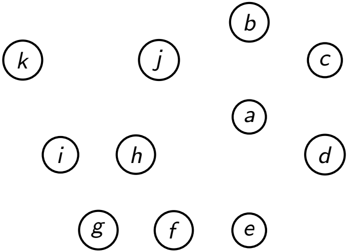
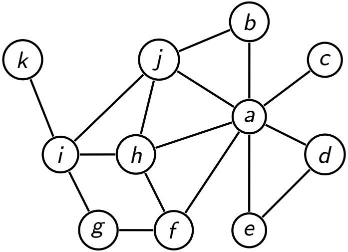
Social Network

agents



Social Network

agents + links



Diffusion Phenomena in Social Networks

Diffusion of a "behaviour":

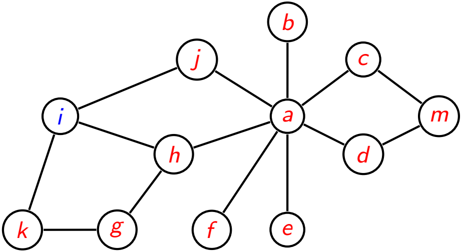
- product
- fashion
- opinion
- infection
- facebook-like
- information
- ...

Threshold Influence

Adopt whenever **enough** network-neighbours have adopted it already.

Example: threshold $= 1/4$

new behaviour old behaviour



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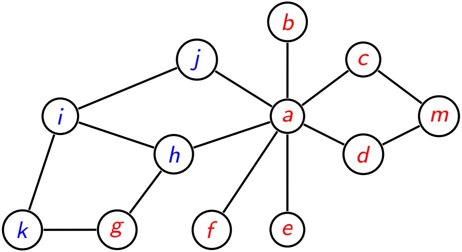

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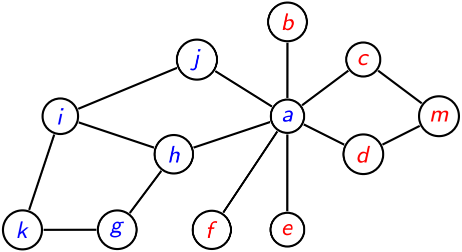

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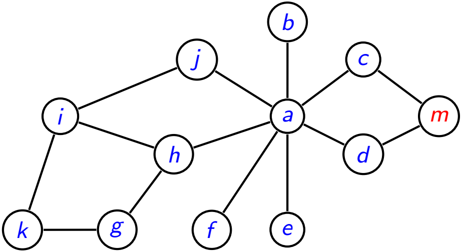

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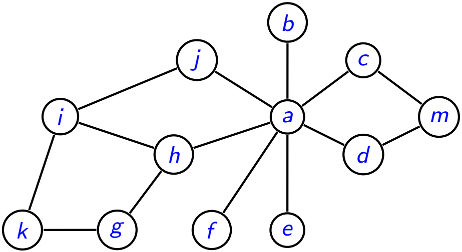

Complete cascade!

Example: threshold $= 1/2$

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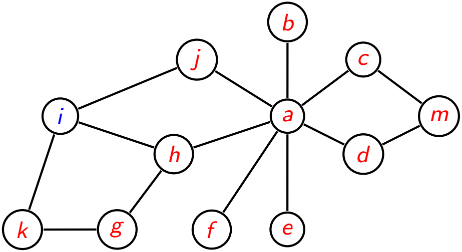

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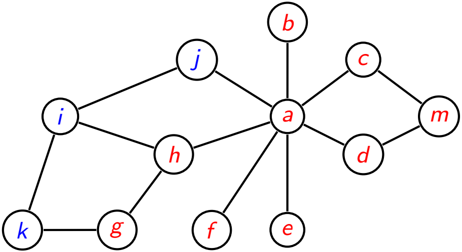

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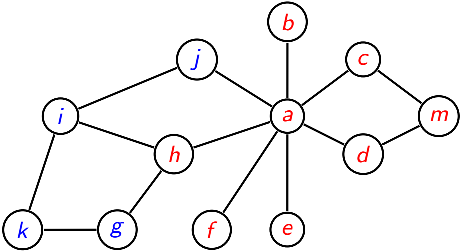

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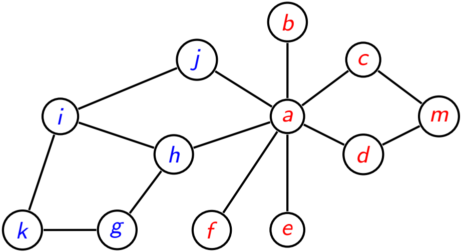

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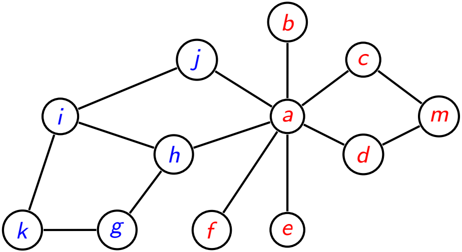

Questions

Does the process stabilize?

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What properties of networks facilitate/slow down diffusion?

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What properties of networks facilitate/slow down diffusion?

When is a cascade complete?

Goal

Model threshold influence using logic:

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1. **Minimal** dynamic logic for **standard** threshold models

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Model threshold influence using logic:

1. **Minimal** dynamic logic for **standard** threshold models
2. **Epistemic** extension for **epistemic** threshold models

Graph

Definition (Network)

A network (or graph) is a pair $(\mathcal{A}, N)$, where:

- $\mathcal{A}$ is a non-empty finite set of *agents*
- $N : \mathcal{A} \to \mathcal{P}(\mathcal{A})$ assigns a set $N(a)$ to each $a \in \mathcal{A}$, such that:
  - $a \notin N(a)$ (Irreflexivity),
  - $b \in N(a)$ if and only if $a \in N(b)$ (Symmetry).

Model

Definition (Threshold model)

A threshold model is a tuple $\mathcal{M} = (\mathcal{A}, N, B, \theta)$ where:

- $(\mathcal{A}, N)$ is a network,
- $B \subseteq \mathcal{A}$ is a *behaviour*
- $\theta \in [0, 1]$ is a uniform *adoption threshold*.

Update

Definition (Threshold model update)

Let $\mathcal{M} = (\mathcal{A}, N, B, \theta)$ be a threshold model.

The updated model $\mathcal{M}' = (\mathcal{A}, N, B', \theta)$, where:

$$B' = B \cup \{a \in \mathcal{A} : \frac{|N(a) \cap B|}{|N(a)|} \geq \theta\}.$$

Minimal Logic

Definition (Threshold influence language $\mathcal{L}_{TI\mathcal{A}^+}$)

$$\varphi := N_a b \mid B_a \mid \neg\varphi \mid \varphi \wedge \varphi \mid [adopt]\varphi$$

where $a, b \in \mathcal{A}^+$ for some finite $\mathcal{A}^+$

Threshold Models

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Threshold Models of bounded size

Models which the language $\mathcal{L}_{TI\mathcal{A}^+}$ can fully describe:

Definition ($\mathcal{A}^+$-threshold model)

An $\mathcal{A}^+$- threshold model is a tuple $\mathcal{M} = (\mathcal{A}, N, B, \theta)$ where:

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Definition ($\mathcal{C}_{\mathcal{A}^+\theta}$)

We denote by $\mathcal{C}_{\mathcal{A}^+\theta}$ the class of all $\mathcal{A}^+$-threshold models with fixed threshold θ.

Semantics

Definition (Truth clauses for $\mathcal{L}_{TI\mathcal{A}^+}$)

Let $\mathcal{M} = (\mathcal{A}, N, B, \theta)$ be an $\mathcal{A}^+$-threshold-model and $N_a b, B_a, \varphi, \psi \in \mathcal{L}_{TI\mathcal{A}^+}$.

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$$\mathcal{M} \models N_a b \text{ iff } b \in N(a)$$

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$\mathcal{M} \vDash N_a b$ iff $b \in N(a)$

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$\mathcal{M} \vDash \neg\varphi$ iff $\mathcal{M} \nvDash \varphi$

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$\mathcal{M} \vDash \neg\varphi$ iff $\mathcal{M} \nvDash \varphi$

$\mathcal{M} \vDash \varphi \wedge \psi$ iff $\mathcal{M} \vDash \varphi$ and $\mathcal{M} \vDash \psi$

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$\mathcal{M} \vDash \varphi \wedge \psi$ iff $\mathcal{M} \vDash \varphi$ and $\mathcal{M} \vDash \psi$

$\mathcal{M} \vDash [adopt]\varphi$ iff $\mathcal{M}' \vDash \varphi$, where $\mathcal{M}' = (\mathcal{A}, N, B', \theta)$, with:

$$B' = B \cup \{a \in \mathcal{A} : \frac{|N(a) \cap B|}{|N(a)|} \geq \theta\}.$$

Axiomatization

Network axioms

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$\neg N_a a$	Irreflexivity
$N_a b \leftrightarrow N_b a$	Symmetry

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$$[adopt]N_a b \leftrightarrow N_a b$$

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$$[adopt]\varphi \wedge \psi \leftrightarrow [adopt]\varphi \wedge [adopt]\psi$$

$$[adopt]B_a \leftrightarrow B_a \vee \bigvee_{\{\mathcal{G}\subseteq\mathcal{N}\subseteq\mathcal{A}^+ : \frac{|\mathcal{G}|}{|\mathcal{N}|}\geq\theta\}} \left(\bigwedge_{b\in\mathcal{N}} N_a b \wedge \bigwedge_{b\notin\mathcal{N}} \neg N_a b \wedge \bigwedge_{b\in\mathcal{G}} B_b\right)$$

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$$[adopt]\neg\varphi \leftrightarrow \neg[adopt]\varphi$$
$$[adopt]\varphi \wedge \psi \leftrightarrow [adopt]\varphi \wedge [adopt]\psi$$
$$[adopt]B_a \leftrightarrow B_a \vee \bigvee_{\{\mathcal{G}\subseteq\mathcal{N}\subseteq\mathcal{A}^+ : \frac{|\mathcal{G}|}{|\mathcal{N}|}\geq\theta\}} \left(\bigwedge_{b\in\mathcal{N}} N_a b \wedge \bigwedge_{b\notin\mathcal{N}} \neg N_a b \wedge \bigwedge_{b\in\mathcal{G}} B_b\right)$$

Theorem (Completeness)

For every $\varphi \in \mathcal{L}_{TI\mathcal{A}^+}$,

$$\models_{\mathcal{C}_{\mathcal{A}^+\theta}} \varphi \text{ iff } \vdash_{L_{TI\mathcal{A}^+\theta}} \varphi$$

Expressing stability?

Stable model

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$$\mathcal{M} \vDash B_a \leftrightarrow [adopt]B_a$$

for all $a \in \mathcal{A}^+$.

Expressing stabilization?

Stabilizing model

For some $n \in \mathbb{N}$:

$$\mathcal{M} \vDash [adopt]^n \varphi \leftrightarrow [adopt]^{n+1} \varphi$$

for all $\varphi \in \mathcal{L}_{TI\mathcal{A}^+}$.

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$$\mathcal{M} \vDash [adopt]^{n}B_a \leftrightarrow [adopt]^{n+1}B_a$$

for all $a \in \mathcal{A}^+$.

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$$\mathcal{M} \vDash [adopt]^n B_a \leftrightarrow [adopt]^{n+1} B_a$$

for all $a \in \mathcal{A}^+$.

All models stabilize in at most $|\mathcal{A}| - 1$ steps

$$\mathcal{M} \vDash [adopt]^{|\mathcal{A}|-1} B_a \leftrightarrow [adopt]^{|\mathcal{A}|} B_a$$

for all $a \in \mathcal{A}^+$.

Clusters

Definition (Cluster of density d)

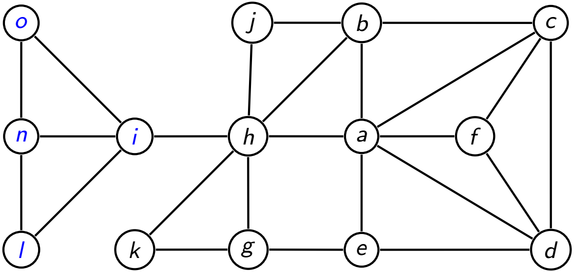
Given a network $(\mathcal{A}, N)$ a cluster of density d is any group $C \subseteq \mathcal{A}$ such that for all $i \in C$,

$$\frac{|N(i) \cap C|}{|N(i)|} \geq d.$$

- Any network contains at least one cluster of density 1, namely $\mathcal{A}$
- and that each singleton $\{a\} \subseteq \mathcal{A}$ is a cluster of density 0 (by irreflexivity).

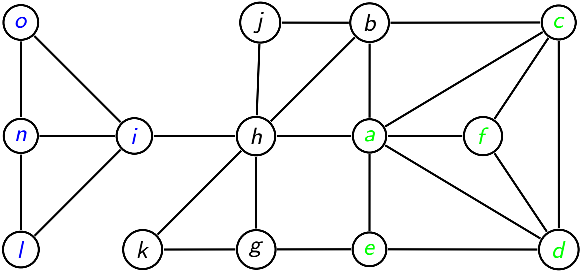
Example: clusters

o,n,i,l : cluster of density 3/4



Example: clusters

o,n,i,l : cluster of density 3/4
c,f,d,a,e: cluster of density 2/3



Clusters and Cascades

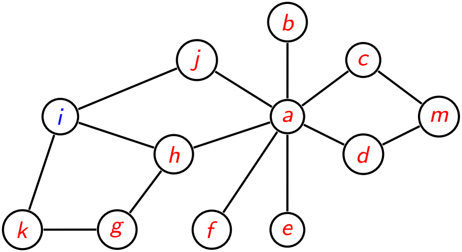
When is a cascade complete (for $\theta \neq 0$)?

When (and only when) there is no cluster of density greater than $1 - \theta$ in $\mathcal{A} \backslash B$

Back to our initial example: threshold $= 1/4$

There is no red cluster of density $> 3/4$

new behaviour old behaviour



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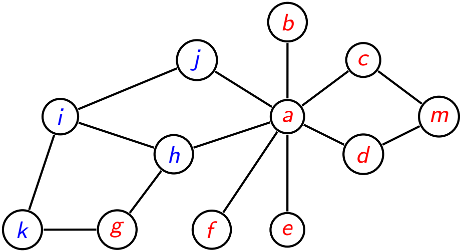

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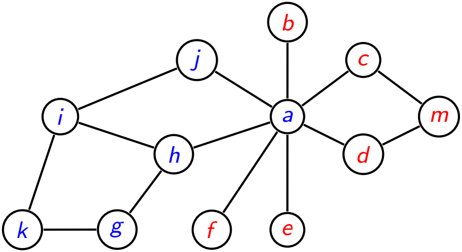

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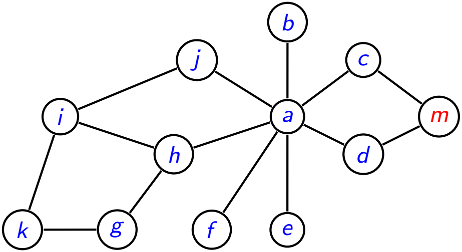

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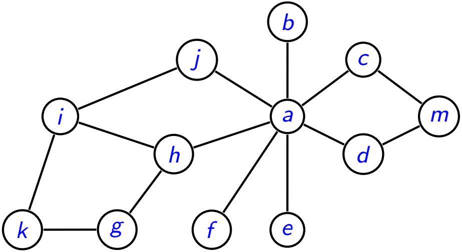

Complete cascade!

Back to our initial example: threshold $= 1/2$

a, b, c, d, m, e, f form a cluster of density $d = 5/7 > 1/2$

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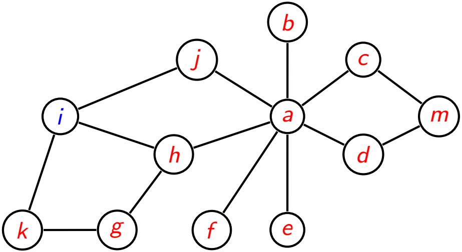

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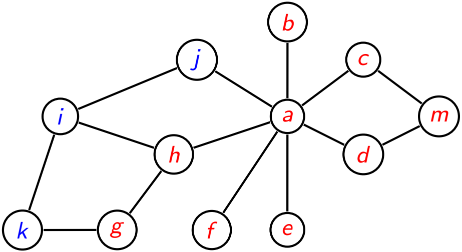

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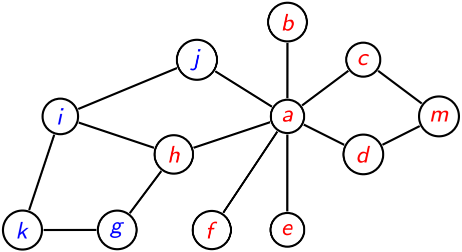

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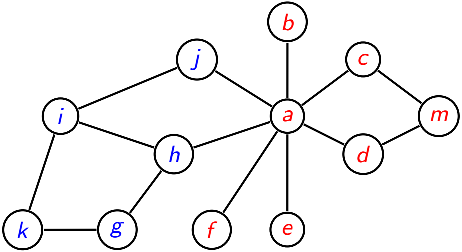

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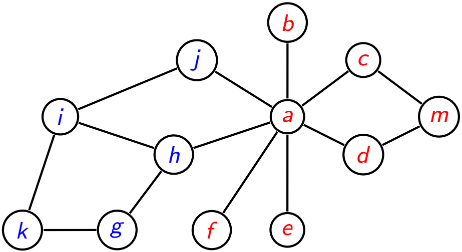

Talking about clusters

$C \subseteq \mathcal{A}$ is a cluster of density d in $(\mathcal{A}, N)$ iff $\mathcal{M} = (\mathcal{A}, N, B, \theta)$ satisfies:

$$\bigwedge_{i \in C} \bigvee_{\{\mathcal{N} \subseteq \mathcal{A} : \frac{|\mathcal{N} \cap C|}{|\mathcal{N}|} \geq d\}} \left(\bigwedge_{j \in \mathcal{N}} N_i j \wedge \bigwedge_{j \notin \mathcal{N}} \neg N_i j\right)$$

Existence of a -cluster

There exists a cluster of density d:

$$\exists C_d := \bigvee_{C \subseteq \mathcal{A}} \bigwedge_{i \in C} \bigvee_{\{\mathcal{N} \subseteq \mathcal{A} : \frac{|\mathcal{N} \cap C|}{|\mathcal{N}|} \geq d\}} (\bigwedge_{j \in \mathcal{N}} N_i j \wedge \bigwedge_{j \notin \mathcal{N}} \neg N_i j)$$

There exists a cluster of density greater than d:

$$\exists C_{>d} := \bigvee_{C \subseteq \mathcal{A}} \bigwedge_{i \in C} \bigvee_{\{\mathcal{N} \subseteq \mathcal{A} : \frac{|\mathcal{N} \cap C|}{|\mathcal{N}|} > d\}} (\bigwedge_{j \in \mathcal{N}} N_i j \wedge \bigwedge_{j \notin \mathcal{N}} \neg N_i j)$$

Existence of a $\neg B$-cluster

There exists a cluster of density d:

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There exists a cluster of density greater than d in which nobody has adopted :

$$\exists C_{>d} \neg B := \bigvee_{C \subseteq \mathcal{A}} \bigwedge_{i \in C} \bigvee_{\{\mathcal{N} \subseteq \mathcal{A}: \frac{|\mathcal{N} \cap C|}{|\mathcal{N}|} > d\}} \left(\bigwedge_{j \in \mathcal{N}} N_i j \wedge \bigwedge_{j \notin \mathcal{N}} \neg N_i j \wedge \neg B_i \right)$$

Clusters and Cascades

A cascade is complete (for $\theta \neq 0$)
iff there is no cluster of density greater than $1 - \theta$ in $\mathcal{A} \backslash B$

$$[adopt]^{|\mathcal{A}|-1} \bigwedge_{i \in \mathcal{A}} B_i \leftrightarrow \neg \exists C_{>(1-\theta)} \neg B.$$

Future research

- Use the minimal logic to prove (new?) things about the threshold influence dynamics.

- Consider less minimal cases:
  - drop constraints on the network structure
  - unadoption (stabilization becomes more interesting),
  - several behaviours,
  - non-uniform thresholds,
  - etc.

Conclusion: Blind influence vs Informed influence?

- SO FAR: Agents react automatically to their environnment.
- BUT WHAT IF... the behaviour of agents depends on what they KNOW/SEE about the behaviour of others around them?

Conclusion: Maybe blind influence is not so blind after all

Under some natural assumptions about the knowledge of agents in a network....

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the dynamics of diffusion among informed agents reduces to the diffusion among our "blind" agents!

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...Stay tuned for Rasmus!

THANK YOU!