

# The Range of Realization

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# Justification Logics (Background)

Justification Logics are like modal logics, except they involve *explicit reasoning* within the language itself.

You will see examples shortly.

Let's start with some history.

Intuitionistic logic  
was intended to be constructive.  
And it is, in a precise sense.

The Brouwer, Heyting, Kolmogorov  
(BHK)  
semantics has a constructive flavor.

It is based on  
an abstract notion of *proof*.

$\perp$  has no proof.

A proof of  $X \wedge Y$  consists of  
a proof of  $X$  and a proof of  $Y$ .

A proof of  $X \vee Y$  consists of  
a proof of  $X$  or a proof of  $Y$   
(and maybe a specification of which one).

A proof of  $X \supset Y$  consists of  
an algorithm converting any proof of  $X$   
into a proof of  $Y$ .

But, what is a proof?

Can this be given an  
arithmetic interpretation?

In 1933 Gödel made a first step.

One can characterize  
intuitionistic “truth”  
using classical validity plus  
informal provability.

Gödel proposed that informal provability should meet the following conditions (writing  $\Box X$  for  $X$  is “provable”).

classical tautologies

$$\Box(A \supset B) \supset (\Box A \supset \Box B)$$

$$\Box A \supset A$$

$$\Box A \supset \Box \Box A$$

$\vdash A$  and  $\vdash A \supset B$  implies  $\vdash B$

$\vdash A$  implies  $\vdash \Box A$

This is the well-known modal logic S4.

Translate intuitionistic formulas  
by putting  $\Box$  before every subformula.

For example,

$$(A \wedge B) \supset A$$

becomes  $\Box((\Box(\Box A \wedge \Box B) \supset \Box A)$

Then,  $X$  is an intuitionistic theorem  
if and only if the translate of  $X$   
is a theorem of S4.

Gödel was an expert on embedding logic into arithmetic.

He noted that S4 does *not* embed into arithmetic.

At least not by using his provability predicate  $(\exists y)(y \text{ is the Gödel number of a proof of } x)$  to interpret  $\Box$ .



In 1938 Gödel had another proposal,  
interpret  $\Box$  as *explicit* provability.

(This moves the existential quantifier to the metalevel.)

This was not published during Gödel's lifetime.

The idea was independently rediscovered  
by Sergei Artemov in the 1990's.

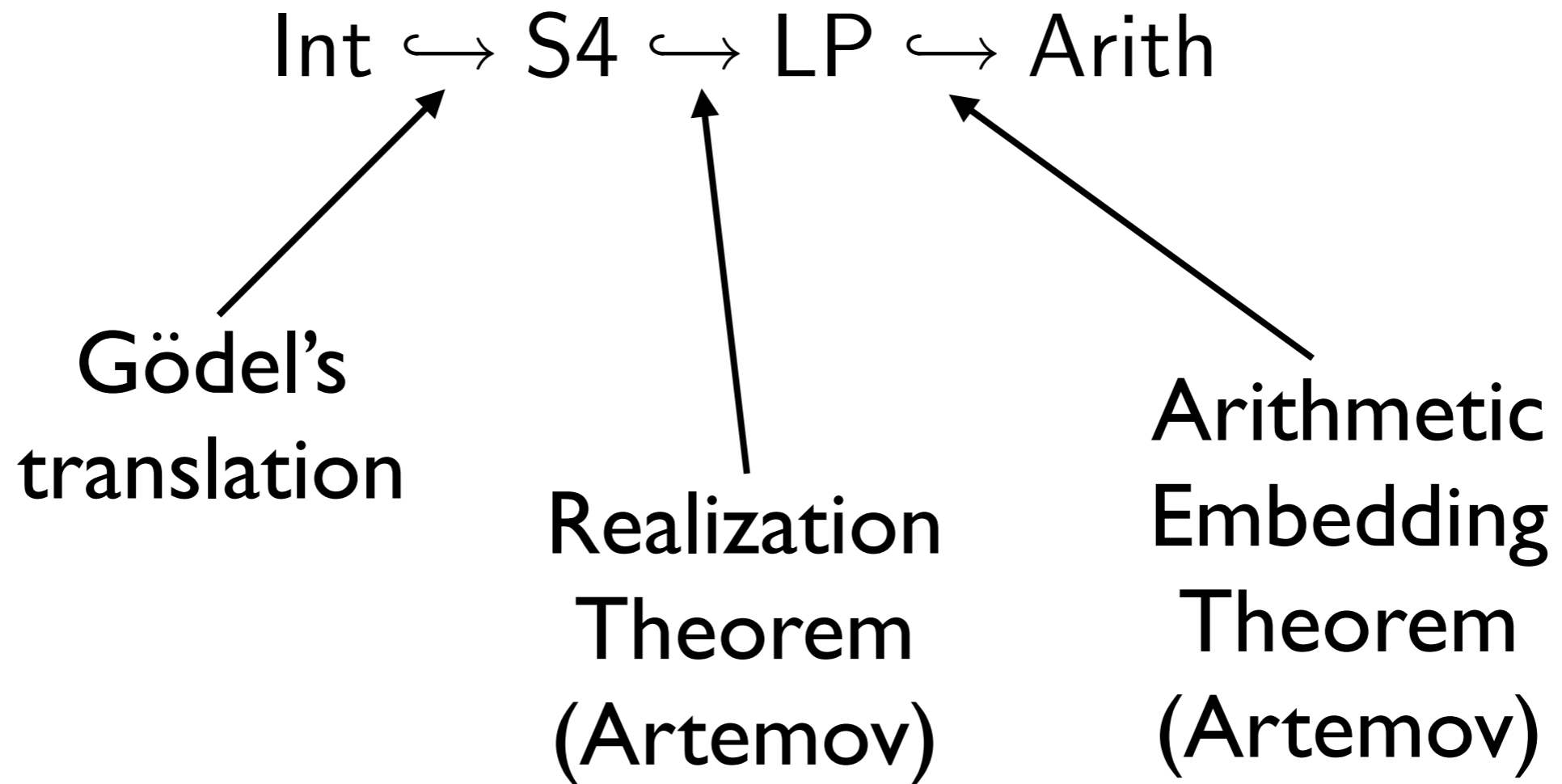
Artemov introduced a logic *LP*  
(logic of proofs)

We will see the details shortly.

It is a kind of *explicit* modal logic.

This will mean something shortly.

# The Basic Picture



Intuitionistic logic has an arithmetic interpretation.

# The Basic Picture

Int  $\hookrightarrow$  S4  $\hookrightarrow$  LP  $\hookrightarrow$  Arith



Realization  
Theorem  
(Artemov)

What we concentrate on.

# And what is LP?

The really new things are  
*proof terms*

(now usually called  
*justification terms*)

Variables,  $v_1, v_2, \dots$  are proof terms.

Constant symbols,  $c_1, c_2, \dots$  are proof terms.

If  $t$  and  $u$  are proof terms, so are  $t + u$  and  $t \cdot u$ .

If  $t$  is a proof term, so is  $!t$ .

Formulas are built up from propositional letters,  $P$ ,  $Q$ ,  $\dots$ , and  $\perp$ .

Using  $\supset$   
and maybe other connectives.

And, if  $t$  is a proof term,  
and  $X$  is a formula,  $t:X$   
is a formula.

Think of  $t:X$  as asserting:  
 $X$  is so, with  $t$  as a proof,  
or  $t$  is a justification for  $X$ .

## The informal ideas:

$t \cdot u$  justifies  $X$  whenever  
 $u$  justifies some formula  $Y$ ,  
and  $t$  justifies  $Y \supset X$ .

$t + u$  justifies  $X$  whenever  
 $t$  justifies  $X$ ,  
or  $u$  justifies  $X$ .

If  $t$  justifies  $X$ ,  
 $!t$  justifies that fact.



Constants justify formulas that  
we do not further analyze;  
that is, axioms.

Variables stand for  
arbitrary justifications.

# LP Axioms

<i>A0.</i>	<b>Classical</b>	Tautologies
<i>A1.</i>	<b>Application</b>	$t:(X \supset Y) \supset (s:X \supset (t \cdot s):Y)$
<i>A2.</i>	<b>Factivity</b>	$t:X \supset X$
<i>A3.</i>	<b>Justification Checker</b>	$t:X \supset !t:(t:X)$
<i>A4.</i>	<b>Weakening</b>	$s:X \supset (s+t):X$ $t:X \supset (s+t):X$

# LP Rules

- R1.* **Modus Ponens**  $\vdash Y$  provided  $\vdash X$  and  $\vdash X \supset Y$
- R2.* **Axiom Necessitation**  $\vdash c:X$  where  $X$  is an axiom  $A0 - A4$  and  $c$  is a justification constant.

Note to self: say something about  
Constant Specifications.

What follows is an  
abbreviated example  
of a proof in LP.

1.  $x:P \supset (x:P \vee y:Q)$
2.  $a:(x:P \supset (x:P \vee y:Q))$  using axiom nec.
3.  $a:(x:P \supset (x:P \vee y:Q)) \supset (!x:x:P \supset [a!\cdot x]:(x:P \vee y:Q))$
4.  $!x:x:P \supset [a!\cdot x]:(x:P \vee y:Q)$
5.  $x:P \supset !x:x:P$
6.  $x:P \supset [a!\cdot x]:(x:P \vee y:Q)$
7.  $y:Q \supset [b!\cdot y]:(x:P \vee y:Q)$  similarly
8.  $x:P \supset [a!\cdot x + b!\cdot y]:(x:P \vee y:Q)$  weakening
9.  $y:Q \supset [a!\cdot x + b!\cdot y]:(x:P \vee y:Q)$  similarly
10.  $(x:P \vee y:Q) \supset [a!\cdot x + b!\cdot y]:(x:P \vee y:Q)$

So we have

$$(x:P \vee y:Q) \supset [a \cdot !x + b \cdot !y]:(x:P \vee y:Q)$$

where  $a$  justifies the tautology

$$x:P \supset (x:P \vee y:Q)$$

and  $b$  justifies the tautology

$$y:Q \supset (x:P \vee y:Q)$$

# Internalization

We have that  
if  $X$  is an axiom,  
 $a:X$  for some constant  $a$ .

In fact, if  $X$  is a theorem,  
 $t:X$  for some justification term  $t$ .

The structure of  $t$   
*internalizes* the proof of  $X$ .

# What is Realization?

For any LP formula  $X$   
let  $X^\circ$  be the result of  
replacing every justification term  
with  $\square$ .

**This is the  
forgetful functor.**



If  $X$  is an LP theorem,  
 $X^\circ$  is an S4 theorem.

True for axioms.

For example,

$$s:(X \supset Y) \supset (t:X \supset [s \cdot t]:Y)$$

becomes

$$\Box(X \supset Y) \supset (\Box X \supset \Box Y).$$

The rules preserve  
this property.

That's all there is  
to this.

But a converse also holds!

If  $X$  is a theorem of S4,  
there is a theorem  $Y$  of LP  
so that  $Y^\circ = X$ .

Better yet,  $Y$  can have  
distinct justification variables  
where  $X$  has negative  $\Box$ .

Positive  $\Box$  occurrences become  
terms computed from these variables.

There is a kind of  
input/output structure.

This is called a  
*normal realization*.

For example, the S4 theorem  
 $(\Box P \vee \Box Q) \supset \Box(\Box P \vee \Box Q)$

has the normal realization

$$(x:P \vee y:Q) \supset [a!\cdot x + b!\cdot y]:(x:P \vee y:Q).$$

S4 can be thought of as  
a logic of knowledge  
(with positive introspection).

Then

$$(KP \vee KQ) \supset K(KP \vee KQ)$$

says something about our  
*implicit* knowledge.

$(x:P \vee y:Q) \supset [a!\cdot!x + b!\cdot!y]:(x:P \vee y:Q)$   
makes reasoning about our knowledge  
*explicit.*

If we have a reason for one of  $P$  or  $Q$   
here is how to go about  
verifying that fact.

# It's a Family!

I've talked about  
modal S4 and justification LP.

But it turns out that  
a large number of modal logics  
have justification counterparts.

Examples are K, T, D, K4, D4.

Since these are sublogics of S4,  
one just omits parts of the  
S4 and LP machinery.

Then counterparts are connected  
via realization theorems.



S5 was an early example  
that needed new justification machinery.

But recently I've realized,  
the family of modal logics  
with justification counterparts  
is very big.

Let's look at S4.2 as an example.

Axiomatically, add to S4,

$$\diamond\Box X \supset \Box\diamond X$$

or equivalently,

$$\Box\neg\Box X \vee \Box\neg\Box\neg X$$

Semantically, use S4 frames  
that are *convergent*.

$uRv_1$  and  $uRv_2$  implies  
there is some  $w$  with  
 $v_1Rw$  and  $v_2Rw$ .

For a justification counterpart,  
add to LP two new function symbols,  
and the axiom

$$f(t, u): \neg t:X \vee g(t, u): \neg u:\neg X$$

Let's call this logic J4.2

Here's an informal  
plausibility argument.

$\neg t:X \vee \neg u:\neg X$   
is provable in LP.

In any context one of the disjuncts must hold.

$f(t, u):\neg t:X \vee g(t, u):\neg u:\neg X$   
says we can compute a justification for  
whichever does hold.

## J4.2 realizes S4.2

For example, here is an S4.2 theorem:

$$[\Diamond\Box A \wedge \Diamond\Box B] \supset \Diamond\Box(A \wedge B)$$

Or equivalently,

$$[\neg\Box\neg\Box A \wedge \neg\Box\neg\Box B] \supset \neg\Box\neg\Box(A \wedge B)$$

It is realized by

$$\{\neg[j_4 \cdot j_3 \cdot !v_5 \cdot g(!v_3, j_2 \cdot v_5 \cdot !v_9)]: \neg v_9:A \wedge \\ \neg[j_5 \cdot f(!v_3, j_2 \cdot v_5 \cdot !v_9)]: \neg v_3:B\} \supset \neg v_5:\neg[j_1 \cdot v_9 \cdot v_3):(A \wedge B)$$

where:

$$j_1:\{A \supset (B \supset (A \wedge B))\}$$

$$j_2:\{\neg[j_1 \cdot v_1 \cdot v_3):(A \wedge B) \supset (v_1:A \supset \neg v_3:B)\}$$

$$j_3:\{v_5:\neg[j_1 \cdot v_1 \cdot v_3):(A \wedge B) \supset \{\neg[j_2 \cdot v_5 \cdot v_2]: \neg v_3:B \supset \neg[v_2 \cdot v_1]:A\}\}$$

$$j_4:\{\neg !v_9:v_9:A \supset \neg v_9:A\}$$

$$j_5:\{\neg !v_3:v_3:B \supset \neg v_3:B\}$$

# The Current State of Things

I'll describe where  
we are so far.

Work is still in progress.



There is a possible world semantics  
for justification logics,  
generally called *Fitting models*.

I'll skip details here.

Axiomatically formulated  
justification logics  
have canonical Fitting models.

Suppose ML is a  
canonical *modal* logic.

Suppose JL is a  
candidate for a  
*justification* counterpart.

Suppose the canonical  
justification model for JL  
is built on a frame for ML.

Then a Realization Theorem  
connects ML and JL.

# What's Missing?

- How do you produce candidates for justification counterparts?
- Are there general conditions for when their canonical models meet the modal frame conditions?
- Are there general conditions for which modal logics this works for? (Sahlqvist perhaps?)

# A Few References

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Thank You